10.1 Conics and Calculus

- **Understand the definition of a conic section.**
- **Analyze and write equations of parabolas using properties of parabolas.**
- **Analyze and write equations of ellipses using properties of ellipses.**
- **Analyze and write equations of hyperbolas using properties of hyperbolas.**

Conic Sections

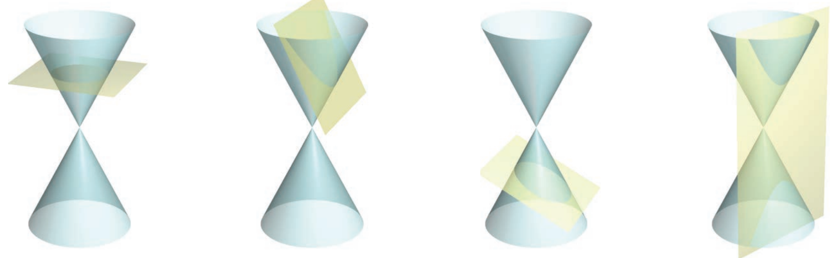
Each **conic section** (or simply **conic**) can be described as the intersection of a plane and a double-napped cone. Notice in Figure 10.1 that for the four basic conics, the intersecting plane does not pass through the vertex of the cone. When the plane passes through the vertex, the resulting figure is a **degenerate conic,** as shown in Figure 10.2.

HYPATIA (370–415 A.D.)

The Greeks discovered conic sections sometime between 600 and 300 B.C. By the beginning of the Alexandrian period, enough was known about conics for Apollonius (262–190 B.C.) to produce an eight-volume work on the subject. Later, toward the end of the Alexandrian period, Hypatia wrote a textbook entitled *On the Conics of Apollonius*. Her death marked the end of major mathematical discoveries in Europe for several hundred years.

The early Greeks were largely concerned with the geometric properties of conics. It was not until 1900 years later, in the early seventeenth century, that the broader applicability of conics became apparent. Conics then played a prominent role in the development of calculus.

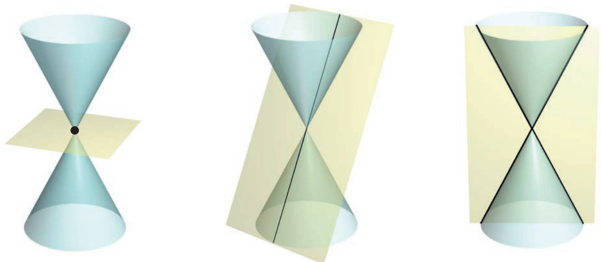
See LarsonCalculus.com to read more of this biography.



Circle Parabola Ellipse Hyperbola

Conic sections

Figure 10.1



Point Line Two intersecting lines

Degenerate conics

Figure 10.2

There are several ways to study conics. You could begin as the Greeks did, by defining the conics in terms of the intersections of planes and cones, or you could define them algebraically in terms of the general second-degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0. \quad \text{General second-degree equation}$$

However, a third approach, in which each of the conics is defined as a **locus** (collection) of points satisfying a certain geometric property, works best. For example, a circle can be defined as the collection of all points (x, y) that are equidistant from a fixed point (h, k). This locus definition easily produces the standard equation of a circle

$$(x - h)^2 + (y - k)^2 = r^2. \quad \text{Standard equation of a circle}$$

For information about rotating second-degree equations in two variables, see Appendix D.

FOR FURTHER INFORMATION To learn more about the mathematical activities of Hypatia, see the article "Hypatia and Her Mathematics" by Michael A. B. Deakin in *The American Mathematical Monthly.* To view this article, go to *MathArticles.com.*

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Parabolas

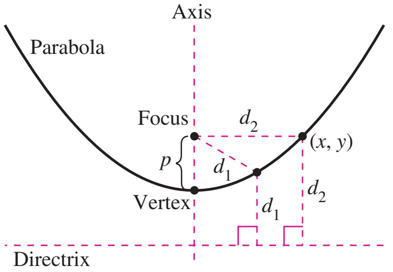


Figure 10.3

A **parabola** is the set of all points (x, y) that are equidistant from a fixed line, the **directrix,** and a fixed point, the **focus,** not on the line. The midpoint between the focus and the directrix is the **vertex,** and the line passing through the focus and the vertex is the **axis** of the parabola. Note in Figure 10.3 that a parabola is symmetric with respect to its axis.

THEOREM 10.1 Standard Equation of a Parabola

The **standard form** of the equation of a parabola with vertex (h, k) and directrix $y = k - p$ is

$$(x - h)^2 = 4p(y - k). \quad \text{Vertical axis}$$

For directrix $x = h - p$, the equation is

$$(y - k)^2 = 4p(x - h). \quad \text{Horizontal axis}$$

The focus lies on the axis p units (*directed distance*) from the vertex. The coordinates of the focus are as follows.

$$(h, k + p) \quad \text{Vertical axis}$$
$$(h + p, k) \quad \text{Horizontal axis}$$

EXAMPLE 1 Finding the Focus of a Parabola

Find the focus of the parabola

$$y = \frac{1}{2} - x - \frac{1}{2}x^2.$$

Solution To find the focus, convert to standard form by completing the square.

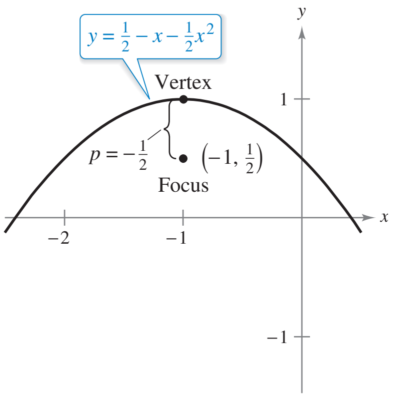
$$y = \frac{1}{2} - x - \frac{1}{2}x^2 \quad \text{Write original equation.}$$
$$2y = 1 - 2x - x^2 \quad \text{Multiply each side by 2.}$$
$$2y = 1 - (x^2 + 2x) \quad \text{Group terms.}$$
$$2y = 2 - (x^2 + 2x + 1) \quad \text{Add and subtract 1 on right side.}$$
$$x^2 + 2x + 1 = -2y + 2$$
$$(x + 1)^2 = -2(y - 1) \quad \text{Write in standard form.}$$

Comparing this equation with

$$(x - h)^2 = 4p(y - k)$$

you can conclude that

$$h = -1, \quad k = 1, \quad \text{and} \quad p = -\frac{1}{2}.$$



Parabola with a vertical axis, $p < 0$
Figure 10.4

Because p is negative, the parabola opens downward, as shown in Figure 10.4. So, the focus of the parabola is p units from the vertex, or

$$(h, k + p) = \left(-1, \frac{1}{2}\right). \quad \text{Focus}$$

A line segment that passes through the focus of a parabola and has endpoints on the parabola is called a **focal chord.** The specific focal chord perpendicular to the axis of the parabola is the **latus rectum.** The next example shows how to determine the length of the latus rectum and the length of the corresponding intercepted arc.

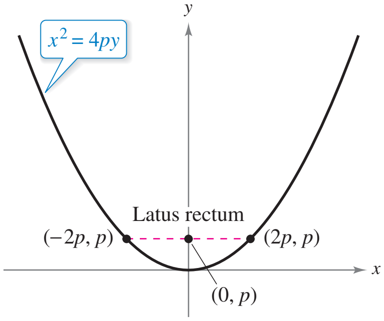
EXAMPLE 2 Focal Chord Length and Arc Length

See LarsonCalculus.com for an interactive version of this type of example.

Find the length of the latus rectum of the parabola

$$x^2 = 4py.$$

Then find the length of the parabolic arc intercepted by the latus rectum.



Length of latus rectum: $4p$
Figure 10.5

Solution Because the latus rectum passes through the focus $(0, p)$ and is perpendicular to the y-axis, the coordinates of its endpoints are

$$(-x, p) \quad \text{and} \quad (x, p).$$

Substituting p for y in the equation of the parabola produces

$$x^2 = 4p(p) \implies x = \pm 2p.$$

So, the endpoints of the latus rectum are $(-2p, p)$ and $(2p, p)$, and you can conclude that its length is $4p$, as shown in Figure 10.5. In contrast, the length of the intercepted arc is

$$s = \int_{-2p}^{2p} \sqrt{1 + (y')^2}\, dx \qquad \text{Use arc length formula.}$$

$$= 2\int_0^{2p} \sqrt{1 + \left(\frac{x}{2p}\right)^2}\, dx \qquad y = \frac{x^2}{4p} \implies y' = \frac{x}{2p}$$

$$= \frac{1}{p}\int_0^{2p} \sqrt{4p^2 + x^2}\, dx \qquad \text{Simplify.}$$

$$= \frac{1}{2p}\left[x\sqrt{4p^2 + x^2} + 4p^2 \ln\left|x + \sqrt{4p^2 + x^2}\right|\right]_0^{2p} \qquad \text{Theorem 8.2}$$

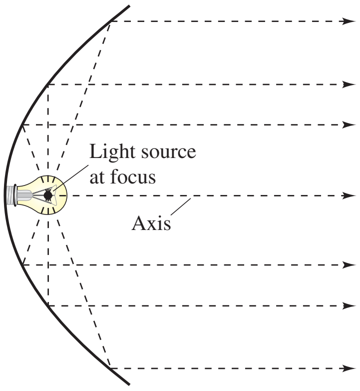
$$= \frac{1}{2p}\left[2p\sqrt{8p^2} + 4p^2 \ln\left(2p + \sqrt{8p^2}\right) - 4p^2 \ln(2p)\right]$$

$$= 2p\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$$

$$\approx 4.59p.$$

One widely used property of a parabola is its reflective property. In physics, a surface is called **reflective** when the tangent line at any point on the surface makes equal angles with an incoming ray and the resulting outgoing ray. The angle corresponding to the incoming ray is the **angle of incidence,** and the angle corresponding to the outgoing ray is the **angle of reflection.** One example of a reflective surface is a flat mirror.

Another type of reflective surface is that formed by revolving a parabola about its axis. The resulting surface has the property that all incoming rays parallel to the axis are directed through the focus of the parabola. This is the principle behind the design of the parabolic mirrors used in reflecting telescopes. Conversely, all light rays emanating from the focus of a parabolic reflector used in a flashlight are parallel, as shown in Figure 10.6.



Parabolic reflector: light is reflected in parallel rays.
Figure 10.6

THEOREM 10.2 Reflective Property of a Parabola

Let P be a point on a parabola. The tangent line to the parabola at point P makes equal angles with the following two lines.

1. The line passing through P and the focus
2. The line passing through P parallel to the axis of the parabola

NICOLAUS COPERNICUS (1473–1543)

Copernicus began to study planetary motion when he was asked to revise the calendar. At that time, the exact length of the year could not be accurately predicted using the theory that Earth was the center of the universe.

See LarsonCalculus.com to read more of this biography.

Ellipses

More than a thousand years after the close of the Alexandrian period of Greek mathematics, Western civilization finally began a Renaissance of mathematical and scientific discovery. One of the principal figures in this rebirth was the Polish astronomer Nicolaus Copernicus. In his work *On the Revolutions of the Heavenly Spheres,* Copernicus claimed that all of the planets, including Earth, revolved about the sun in circular orbits. Although some of Copernicus's claims were invalid, the controversy set off by his heliocentric theory motivated astronomers to search for a mathematical model to explain the observed movements of the sun and planets. The first to find an accurate model was the German astronomer Johannes Kepler (1571–1630). Kepler discovered that the planets move about the sun in elliptical orbits, with the sun not as the center but as a focal point of the orbit.

The use of ellipses to explain the movements of the planets is only one of many practical and aesthetic uses. As with parabolas, you will begin your study of this second type of conic by defining it as a locus of points. Now, however, *two* focal points are used rather than one.

An **ellipse** is the set of all points (x, y) the sum of whose distances from two distinct fixed points called **foci** is constant. (See Figure 10.7.) The line through the foci intersects the ellipse at two points, called the **vertices.** The chord joining the vertices is the **major axis,** and its midpoint is the **center** of the ellipse. The chord perpendicular to the major axis at the center is the **minor axis** of the ellipse. (See Figure 10.8.)

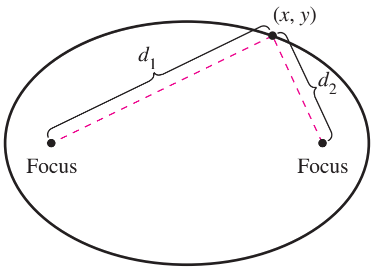


Figure 10.7

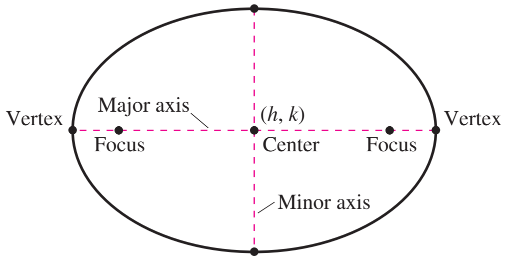


Figure 10.8

THEOREM 10.3 Standard Equation of an Ellipse

The standard form of the equation of an ellipse with center (h, k) and major and minor axes of lengths $2a$ and $2b$, where $a > b$, is

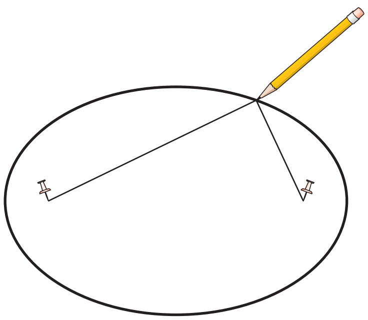
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \qquad \text{Major axis is horizontal.}$$

or

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1. \qquad \text{Major axis is vertical.}$$

The foci lie on the major axis, c units from the center, with

$$c^2 = a^2 - b^2.$$



If the ends of a fixed length of string are fastened to the thumbtacks and the string is drawn taut with a pencil, then the path traced by the pencil will be an ellipse.

Figure 10.9

You can visualize the definition of an ellipse by imagining two thumbtacks placed at the foci, as shown in Figure 10.9.

FOR FURTHER INFORMATION To learn about how an ellipse may be "exploded" into a parabola, see the article "Exploding the Ellipse" by Arnold Good in *Mathematics Teacher.* To view this article, go to *MathArticles.com.*

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EXAMPLE 3 Analyzing an Ellipse

Find the center, vertices, and foci of the ellipse

$$4x^2 + y^2 - 8x + 4y - 8 = 0. \qquad \text{General second-degree equation}$$

Solution By completing the square, you can write the original equation in standard form.

$$4x^2 + y^2 - 8x + 4y - 8 = 0 \qquad \text{Write original equation.}$$
$$4x^2 - 8x + y^2 + 4y = 8$$
$$4(x^2 - 2x + 1) + (y^2 + 4y + 4) = 8 + 4 + 4$$
$$4(x - 1)^2 + (y + 2)^2 = 16$$
$$\frac{(x - 1)^2}{4} + \frac{(y + 2)^2}{16} = 1 \qquad \text{Write in standard form.}$$

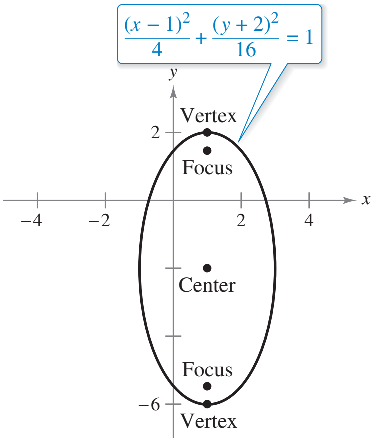
So, the major axis is parallel to the y-axis, where $h = 1$, $k = -2$, $a = 4$, $b = 2$, and $c = \sqrt{16 - 4} = 2\sqrt{3}$. So, you obtain the following.

Center: $(1, -2)$ — (h, k)

Vertices: $(1, -6)$ and $(1, 2)$ — $(h, k \pm a)$

Foci: $(1, -2 - 2\sqrt{3})$ and $(1, -2 + 2\sqrt{3})$ — $(h, k \pm c)$

The graph of the ellipse is shown in Figure 10.10.



Ellipse with a vertical major axis.
Figure 10.10

In Example 3, the constant term in the general second-degree equation is $F = -8$. For a constant term greater than or equal to 8, you would have obtained one of the degenerate cases shown below.

1. $F = 8$, single point, $(1, -2)$: $\dfrac{(x - 1)^2}{4} + \dfrac{(y + 2)^2}{16} = 0$
2. $F > 8$, no solution points: $\dfrac{(x - 1)^2}{4} + \dfrac{(y + 2)^2}{16} < 0$

EXAMPLE 4 The Orbit of the Moon

The moon orbits Earth in an elliptical path with the center of Earth at one focus, as shown in Figure 10.11. The major and minor axes of the orbit have lengths of 768,800 kilometers and 767,640 kilometers, respectively. Find the greatest and least distances (the apogee and perigee) from Earth's center to the moon's center.

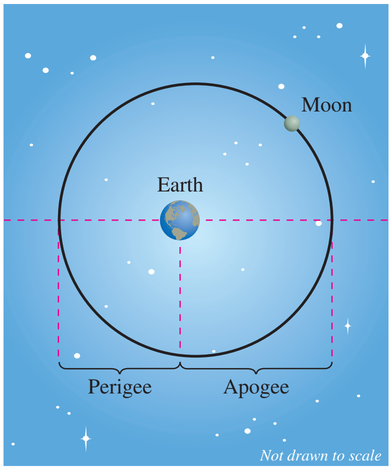


Figure 10.11

Solution Begin by solving for a and b.

$$2a = 768{,}800 \qquad \text{Length of major axis}$$
$$a = 384{,}400 \qquad \text{Solve for } a.$$
$$2b = 767{,}640 \qquad \text{Length of minor axis}$$
$$b = 383{,}820 \qquad \text{Solve for } b.$$

Now, using these values, you can solve for c as follows.

$$c = \sqrt{a^2 - b^2} \approx 21{,}108$$

The greatest distance between the center of Earth and the center of the moon is

$$a + c \approx 405{,}508 \text{ kilometers}$$

and the least distance is

$$a - c \approx 363{,}292 \text{ kilometers.}$$

FOR FURTHER INFORMATION For more information on some uses of the reflective properties of conics, see the article "Parabolic Mirrors, Elliptic and Hyperbolic Lenses" by Mohsen Maesumi in *The American Mathematical Monthly*. Also see the article "The Geometry of Microwave Antennas" by William R. Parzynski in *Mathematics Teacher.*

Theorem 10.2 presented a reflective property of parabolas. Ellipses have a similar reflective property. You are asked to prove the next theorem in Exercise 84.

THEOREM 10.4 Reflective Property of an Ellipse

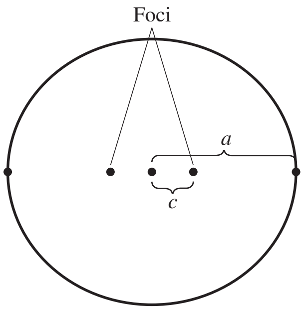
Let P be a point on an ellipse. The tangent line to the ellipse at point P makes equal angles with the lines through P and the foci.

One of the reasons that astronomers had difficulty detecting that the orbits of the planets are ellipses is that the foci of the planetary orbits are relatively close to the center of the sun, making the orbits nearly circular. To measure the ovalness of an ellipse, you can use the concept of **eccentricity.**

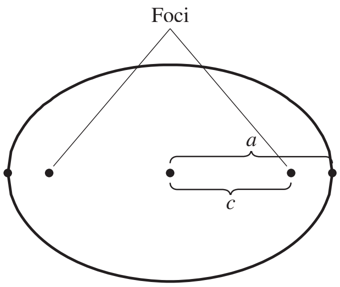
Definition of Eccentricity of an Ellipse

The **eccentricity** e of an ellipse is given by the ratio

$$e = \frac{c}{a}.$$



(a) $\frac{c}{a}$ is small.



(b) $\frac{c}{a}$ is close to 1.

Eccentricity is the ratio $\frac{c}{a}$.

Figure 10.12

To see how this ratio is used to describe the shape of an ellipse, note that because the foci of an ellipse are located along the major axis between the vertices and the center, it follows that

$$0 < c < a.$$

For an ellipse that is nearly circular, the foci are close to the center and the ratio c/a is close to 0, and for an elongated ellipse, the foci are close to the vertices and the ratio c/a is close to 1, as shown in Figure 10.12. Note that

$$0 < e < 1$$

for every ellipse.

The orbit of the moon has an eccentricity of $e \approx 0.0549$, and the eccentricities of the eight planetary orbits are listed below.

Mercury:	$e \approx 0.2056$	Jupiter:	$e \approx 0.0484$
Venus:	$e \approx 0.0068$	Saturn:	$e \approx 0.0542$
Earth:	$e \approx 0.0167$	Uranus:	$e \approx 0.0472$
Mars:	$e \approx 0.0934$	Neptune:	$e \approx 0.0086$

You can use integration to show that the area of an ellipse is $A = \pi ab$. For instance, the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is

$$A = 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

$$= \frac{4b}{a}\int_0^{\pi/2} a^2 \cos^2 \theta \, d\theta. \qquad \text{Trigonometric substitution } x = a \sin \theta$$

However, it is not so simple to find the *circumference* of an ellipse. The next example shows how to use eccentricity to set up an "elliptic integral" for the circumference of an ellipse.

EXAMPLE 5 Finding the Circumference of an Ellipse

See LarsonCalculus.com for an interactive version of this type of example.

Show that the circumference of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ is

$$4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,d\theta. \qquad e = \frac{c}{a}$$

Solution Because the ellipse is symmetric with respect to both the x-axis and the y-axis, you know that its circumference C is four times the arc length of

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

in the first quadrant. The function y is differentiable for all x in the interval $[0, a]$ except at $x = a$. So, the circumference is given by the improper integral

$$C = \lim_{d\to a^-} 4\int_0^d \sqrt{1 + (y')^2}\,dx = 4\int_0^a \sqrt{1 + (y')^2}\,dx = 4\int_0^a \sqrt{1 + \frac{b^2x^2}{a^2(a^2 - x^2)}}\,dx.$$

Using the trigonometric substitution $x = a\sin\theta$, you obtain

$$\begin{aligned} C &= 4\int_0^{\pi/2}\sqrt{1 + \frac{b^2\sin^2\theta}{a^2\cos^2\theta}}(a\cos\theta)\,d\theta \\ &= 4\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta \\ &= 4\int_0^{\pi/2}\sqrt{a^2(1 - \sin^2\theta) + b^2\sin^2\theta}\,d\theta \\ &= 4\int_0^{\pi/2}\sqrt{a^2 - (a^2 - b^2)\sin^2\theta}\,d\theta. \end{aligned}$$

Because $e^2 = c^2/a^2 = (a^2 - b^2)/a^2$, you can rewrite this integral as

$$C = 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}\,d\theta.$$

AREA AND CIRCUMFERENCE OF AN ELLIPSE

In his work with elliptic orbits in the early 1600's, Johannes Kepler successfully developed a formula for the area of an ellipse, $A = \pi ab$. He was less successful, however, in developing a formula for the circumference of an ellipse; the best he could do was to give the approximate formula $C = \pi(a + b)$.

A great deal of time has been devoted to the study of elliptic integrals. Such integrals generally do not have elementary antiderivatives. To find the circumference of an ellipse, you must usually resort to an approximation technique.

EXAMPLE 6 Approximating the Value of an Elliptic Integral

Use the elliptic integral in Example 5 to approximate the circumference of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{16} = 1.$$

Solution Because $e^2 = c^2/a^2 = (a^2 - b^2)/a^2 = 9/25$, you have

$$C = (4)(5)\int_0^{\pi/2}\sqrt{1 - \frac{9\sin^2\theta}{25}}\,d\theta.$$

Applying Simpson's Rule with $n = 4$ produces

$$\begin{aligned} C &\approx 20\left(\frac{\pi}{6}\right)\left(\frac{1}{4}\right)[1 + 4(0.9733) + 2(0.9055) + 4(0.8323) + 0.8] \\ &\approx 28.36. \end{aligned}$$

So, the ellipse has a circumference of about 28.36 units, as shown in Figure 10.13.

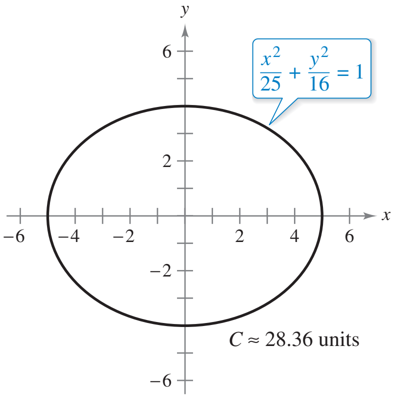


Figure 10.13

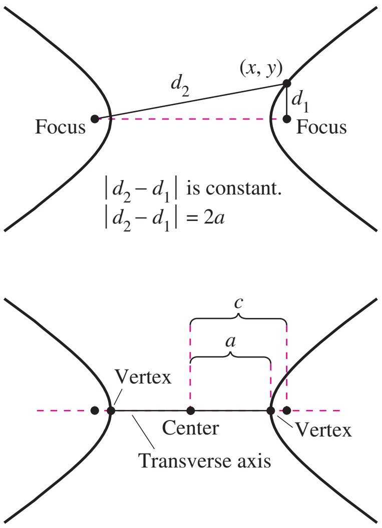


Figure 10.14

Hyperbolas

The definition of a hyperbola is similar to that of an ellipse. For an ellipse, the *sum* of the distances between the foci and a point on the ellipse is fixed, whereas for a hyperbola, the absolute value of the *difference* between these distances is fixed.

A **hyperbola** is the set of all points (x, y) for which the absolute value of the difference between the distances from two distinct fixed points called **foci** is constant. (See Figure 10.14.) The line through the two foci intersects a hyperbola at two points called the **vertices.** The line segment connecting the vertices is the **transverse axis,** and the midpoint of the transverse axis is the **center** of the hyperbola. One distinguishing feature of a hyperbola is that its graph has two separate *branches.*

THEOREM 10.5 Standard Equation of a Hyperbola

The standard form of the equation of a hyperbola with center at (h, k) is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \qquad \text{Transverse axis is horizontal.}$$

or

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1. \qquad \text{Transverse axis is vertical.}$$

The vertices are a units from the center, and the foci are c units from the center, where $c^2 = a^2 + b^2$.

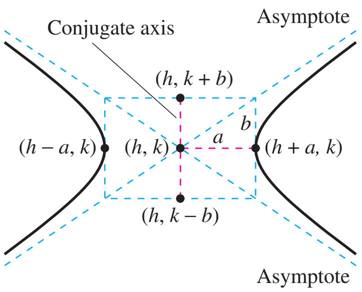


Figure 10.15

Note that the constants a, b, and c do not have the same relationship for hyperbolas as they do for ellipses. For hyperbolas, $c^2 = a^2 + b^2$, but for ellipses, $c^2 = a^2 - b^2$.

An important aid in sketching the graph of a hyperbola is the determination of its **asymptotes,** as shown in Figure 10.15. Each hyperbola has two asymptotes that intersect at the center of the hyperbola. The asymptotes pass through the vertices of a rectangle of dimensions $2a$ by $2b$, with its center at (h, k). The line segment of length $2b$ joining

$$(h, k + b)$$

and

$$(h, k - b)$$

is referred to as the **conjugate axis** of the hyperbola.

THEOREM 10.6 Asymptotes of a Hyperbola

For a *horizontal* transverse axis, the equations of the asymptotes are

$$y = k + \frac{b}{a}(x - h) \quad \text{and} \quad y = k - \frac{b}{a}(x - h).$$

For a *vertical* transverse axis, the equations of the asymptotes are

$$y = k + \frac{a}{b}(x - h) \quad \text{and} \quad y = k - \frac{a}{b}(x - h).$$

In Figure 10.15, you can see that the asymptotes coincide with the diagonals of the rectangle with dimensions $2a$ and $2b$, centered at (h, k). This provides you with a quick means of sketching the asymptotes, which in turn aids in sketching the hyperbola.

EXAMPLE 7 Using Asymptotes to Sketch a Hyperbola

See LarsonCalculus.com for an interactive version of this type of example.

Sketch the graph of the hyperbola

$$4x^2 - y^2 = 16.$$

Solution Begin by rewriting the equation in standard form.

$$\frac{x^2}{4} - \frac{y^2}{16} = 1$$

The transverse axis is horizontal and the vertices occur at $(-2, 0)$ and $(2, 0)$. The ends of the conjugate axis occur at $(0, -4)$ and $(0, 4)$. Using these four points, you can sketch the rectangle shown in Figure 10.16(a). By drawing the asymptotes through the corners of this rectangle, you can complete the sketch as shown in Figure 10.16(b).

TECHNOLOGY You can use a graphing utility to verify the graph obtained in Example 7 by solving the original equation for y and graphing the following equations.

$$y_1 = \sqrt{4x^2 - 16}$$
$$y_2 = -\sqrt{4x^2 - 16}$$

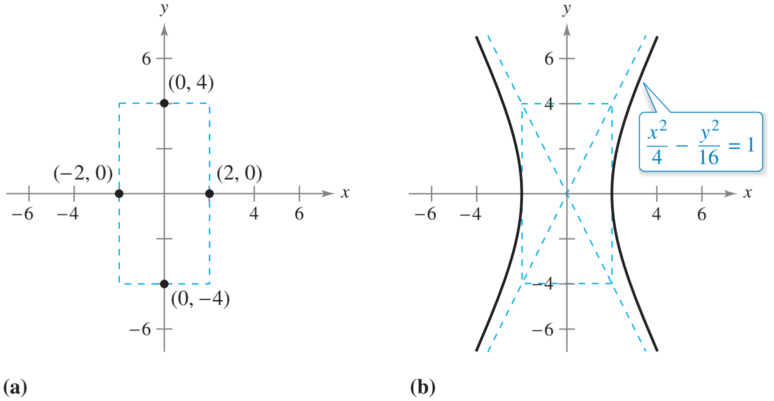


Figure 10.16

Definition of Eccentricity of a Hyperbola

The **eccentricity** e of a hyperbola is given by the ratio

$$e = \frac{c}{a}.$$

FOR FURTHER INFORMATION To read about using a string that traces both elliptic and hyperbolic arcs having the same foci, see the article "Ellipse to Hyperbola: 'With This String I Thee Wed'" by Tom M. Apostol and Mamikon A. Mnatsakanian in *Mathematics Magazine*. To view this article, go to *MathArticles.com*.

As with an ellipse, the **eccentricity** of a hyperbola is $e = c/a$. Because $c > a$ for hyperbolas, it follows that $e > 1$ for hyperbolas. If the eccentricity is large, then the branches of the hyperbola are nearly flat. If the eccentricity is close to 1, then the branches of the hyperbola are more pointed, as shown in Figure 10.17.

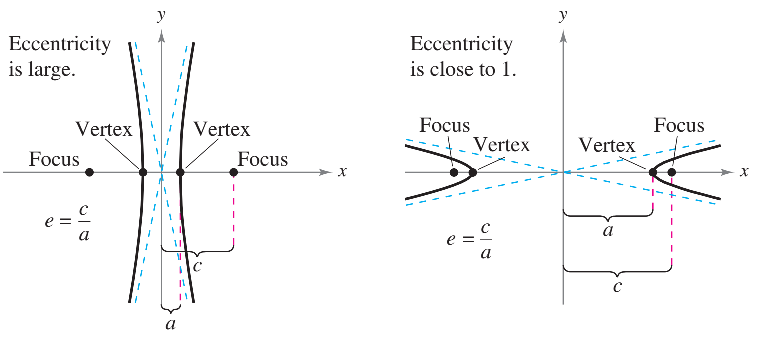
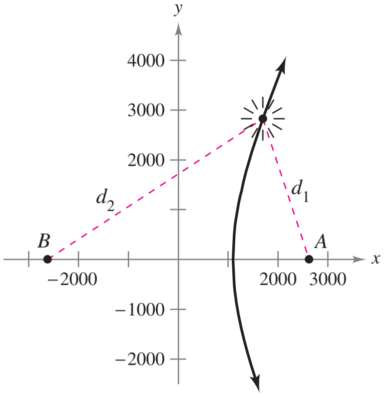


Figure 10.17

The application in Example 8 was developed during World War II. It shows how the properties of hyperbolas can be used in radar and other detection systems.

EXAMPLE 8 A Hyperbolic Detection System

Two microphones, 1 mile apart, record an explosion. Microphone A receives the sound 2 seconds before microphone B. Where was the explosion?



$2c = 5280$
$d_2 - d_1 = 2a = 2200$
Figure 10.18

Solution Assuming that sound travels at 1100 feet per second, you know that the explosion took place 2200 feet farther from B than from A, as shown in Figure 10.18. The locus of all points that are 2200 feet closer to A than to B is one branch of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where

$$c = \frac{1 \text{ mile}}{2} = \frac{5280 \text{ ft}}{2} = 2640 \text{ feet}$$

and

$$a = \frac{2200 \text{ ft}}{2} = 1100 \text{ feet.}$$

Because $c^2 = a^2 + b^2$, it follows that

$$\begin{aligned} b^2 &= c^2 - a^2 \\ &= (2640)^2 - (1100)^2 \\ &= 5{,}759{,}600 \end{aligned}$$

and you can conclude that the explosion occurred somewhere on the right branch of the hyperbola

$$\frac{x^2}{1{,}210{,}000} - \frac{y^2}{5{,}759{,}600} = 1.$$

In Example 8, you were able to determine only the hyperbola on which the explosion occurred, but not the exact location of the explosion. If, however, you had received the sound at a third position C, then two other hyperbolas would be determined. The exact location of the explosion would be the point at which these three hyperbolas intersect.

Another interesting application of conics involves the orbits of comets in our solar system. Of the 610 comets identified prior to 1970, 245 have elliptical orbits, 295 have parabolic orbits, and 70 have hyperbolic orbits. The center of the sun is a focus of each orbit, and each orbit has a vertex at the point at which the comet is closest to the sun. Undoubtedly, many comets with parabolic or hyperbolic orbits have not been identified—such comets pass through our solar system only once. Only comets with elliptical orbits, such as Halley's comet, remain in our solar system.

The type of orbit for a comet can be determined as follows.

1. Ellipse: $v < \sqrt{2GM/p}$
2. Parabola: $v = \sqrt{2GM/p}$
3. Hyperbola: $v > \sqrt{2GM/p}$

In each of the above, p is the distance between one vertex and one focus of the comet's orbit (in meters), v is the velocity of the comet at the vertex (in meters per second), $M \approx 1.989 \times 10^{30}$ kilograms is the mass of the sun, and $G \approx 6.67 \times 10^{-8}$ cubic meters per kilogram-second squared is the gravitational constant.

CAROLINE HERSCHEL (1750–1848)

The first woman to be credited with detecting a new comet was the English astronomer Caroline Herschel. During her life, Caroline Herschel discovered a total of eight new comets.
See LarsonCalculus.com to read more of this biography.

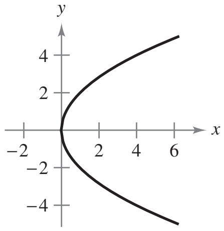
Caroline Herschel (1750-1848), 1829, Tielemans, Martin Francois (1784-1864)/Private Collection/The Bridgeman Art Library

10.1 Exercises

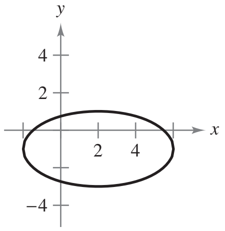
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Matching **In Exercises 1–6, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

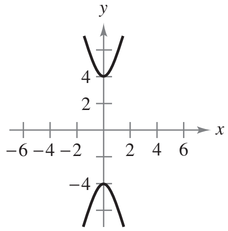
(a)



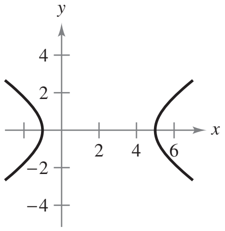
(b)



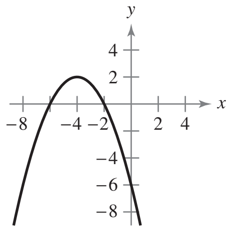
(c)



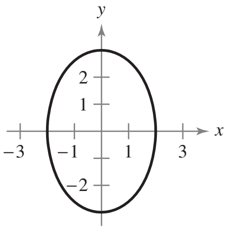
(d)



(e)



(f)



1. $y^2 = 4x$
2. $(x + 4)^2 = -2(y - 2)$
3. $\dfrac{y^2}{16} - \dfrac{x^2}{1} = 1$
4. $\dfrac{(x-2)^2}{16} + \dfrac{(y+1)^2}{4} = 1$
5. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$
6. $\dfrac{(x-2)^2}{9} - \dfrac{y^2}{4} = 1$

Sketching a Parabola **In Exercises 7–14, find the vertex, focus, and directrix of the parabola, and sketch its graph.**

7. $y^2 = -8x$
8. $x^2 + 6y = 0$
9. $(x + 5) + (y - 3)^2 = 0$
10. $(x - 6)^2 + 8(y + 7) = 0$
11. $y^2 - 4y - 4x = 0$
12. $y^2 + 6y + 8x + 25 = 0$
13. $x^2 + 4x + 4y - 4 = 0$
14. $y^2 + 4y + 8x - 12 = 0$

Finding an Equation of a Parabola **In Exercises 15–22, find an equation of the parabola.**

15. Vertex: $(5, 4)$
    Focus: $(3, 4)$
16. Vertex: $(-2, 1)$
    Focus: $(-2, -1)$
17. Vertex: $(0, 5)$
    Directrix: $y = -3$
18. Focus: $(2, 2)$
    Directrix: $x = -2$
19. Vertex: $(0, 4)$
    Points on the parabola: $(-2, 0), (2, 0)$
20. Vertex: $(2, 4)$
    Points on the parabola: $(0, 0), (4, 0)$
21. Axis is parallel to y-axis; graph passes through $(0, 3)$, $(3, 4)$, and $(4, 11)$.
22. Directrix: $y = -2$; endpoints of latus rectum are $(0, 2)$ and $(8, 2)$.

Sketching an Ellipse **In Exercises 23–28, find the center, foci, vertices, and eccentricity of the ellipse, and sketch its graph.**

23. $16x^2 + y^2 = 16$
24. $3x^2 + 7y^2 = 63$
25. $\dfrac{(x-3)^2}{16} + \dfrac{(y-1)^2}{25} = 1$
26. $(x + 4)^2 + \dfrac{(y+6)^2}{1/4} = 1$
27. $9x^2 + 4y^2 + 36x - 24y + 36 = 0$
28. $16x^2 + 25y^2 - 64x + 150y + 279 = 0$

Finding an Equation of an Ellipse **In Exercises 29–34, find an equation of the ellipse.**

29. Center: $(0, 0)$
    Focus: $(5, 0)$
    Vertex: $(6, 0)$
30. Vertices: $(0, 3), (8, 3)$
    Eccentricity: $\frac{3}{4}$
31. Vertices: $(3, 1), (3, 9)$
    Minor axis length: 6
32. Foci: $(0, \pm 9)$
    Major axis length: 22
33. Center: $(0, 0)$
    Major axis: horizontal
    Points on the ellipse: $(3, 1), (4, 0)$
34. Center: $(1, 2)$
    Major axis: vertical
    Points on the ellipse: $(1, 6), (3, 2)$

Sketching a Hyperbola **In Exercises 35–40, find the center, foci, and vertices of the hyperbola, and sketch its graph using asymptotes as an aid.**

35. $\dfrac{x^2}{25} - \dfrac{y^2}{16} = 1$
36. $\dfrac{(y+3)^2}{225} - \dfrac{(x-5)^2}{64} = 1$
37. $9x^2 - y^2 - 36x - 6y + 18 = 0$
38. $y^2 - 16x^2 + 64x - 208 = 0$
39. $x^2 - 9y^2 + 2x - 54y - 80 = 0$
40. $9x^2 - 4y^2 + 54x + 8y + 78 = 0$

Finding an Equation of a Hyperbola **In Exercises 41–48, find an equation of the hyperbola.**

41. Vertices: $(\pm 1, 0)$
    Asymptotes: $y = \pm 5x$
42. Vertices: $(0, \pm 4)$
    Asymptotes: $y = \pm 2x$
43. Vertices: $(2, \pm 3)$
    Point on graph: $(0, 5)$
44. Vertices: $(2, \pm 3)$
    Foci: $(2, \pm 5)$
45. Center: $(0, 0)$
    Vertex: $(0, 2)$
    Focus: $(0, 4)$
46. Center: $(0, 0)$
    Vertex: $(6, 0)$
    Focus: $(10, 0)$
47. Vertices: $(0, 2), (6, 2)$
    Asymptotes: $y = \frac{2}{3}x$
    $y = 4 - \frac{2}{3}x$
48. Focus: $(20, 0)$
    Asymptotes: $y = \pm\frac{3}{4}x$

Finding Equations of Tangent Lines and Normal Lines

In Exercises 49 and 50, find equations for (a) the tangent lines and (b) the normal lines to the hyperbola for the given value of x.

49. $\frac{x^2}{9} - y^2 = 1, \quad x = 6$

50. $\frac{y^2}{4} - \frac{x^2}{2} = 1, \quad x = 4$

Classifying the Graph of an Equation

In Exercises 51–58, classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

51. $x^2 + 4y^2 - 6x + 16y + 21 = 0$

52. $4x^2 - y^2 - 4x - 3 = 0$

53. $25x^2 - 10x - 200y - 119 = 0$

54. $y^2 - 4y = x + 5$

55. $9x^2 + 9y^2 - 36x + 6y + 34 = 0$

56. $2x(x - y) = y(3 - y - 2x)$

57. $3(x - 1)^2 = 6 + 2(y + 1)^2$

58. $9(x + 3)^2 = 36 - 4(y - 2)^2$

WRITING ABOUT CONCEPTS

59. Parabola

(a) Give the definition of a parabola.

(b) Give the standard forms of a parabola with vertex at (h, k).

(c) In your own words, state the reflective property of a parabola.

60. Ellipse

(a) Give the definition of an ellipse.

(b) Give the standard form of an ellipse with center at (h, k).

61. Hyperbola

(a) Give the definition of a hyperbola.

(b) Give the standard forms of a hyperbola with center at (h, k).

(c) Write equations for the asymptotes of a hyperbola.

62. Eccentricity Define the eccentricity of an ellipse. In your own words, describe how changes in the eccentricity affect the ellipse.

63. Using an Equation Consider the equation

$$9x^2 + 4y^2 - 36x - 24y - 36 = 0.$$

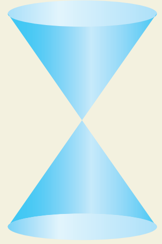
(a) Classify the graph of the equation as a circle, a parabola, an ellipse, or a hyperbola.

(b) Change the $4y^2$-term in the equation to $-4y^2$. Classify the graph of the new equation.

(c) Change the $9x^2$-term in the original equation to $4x^2$. Classify the graph of the new equation.

(d) Describe one way you could change the original equation so that its graph is a parabola.

64. HOW DO YOU SEE IT? In parts (a)–(d), describe in words how a plane could intersect with the double-napped cone to form the conic section (see figure).



(a) Circle (b) Ellipse

(c) Parabola (d) Hyperbola

65. Solar Collector A solar collector for heating water is constructed with a sheet of stainless steel that is formed into the shape of a parabola (see figure). The water will flow through a pipe that is located at the focus of the parabola. At what distance from the vertex is the pipe?

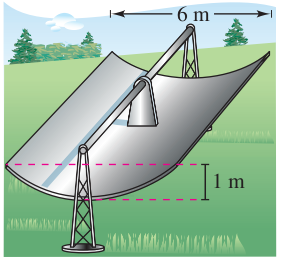


Figure for 65

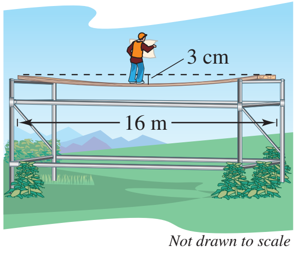


Figure for 66

66. Beam Deflection A simply supported beam that is 16 meters long has a load concentrated at the center (see figure). The deflection of the beam at its center is 3 centimeters. Assume that the shape of the deflected beam is parabolic.

(a) Find an equation of the parabola. (Assume that the origin is at the center of the beam.)

(b) How far from the center of the beam is the deflection 1 centimeter?

67. Proof

(a) Prove that any two distinct tangent lines to a parabola intersect.

(b) Demonstrate the result of part (a) by finding the point of intersection of the tangent lines to the parabola $x^2 - 4x - 4y = 0$ at the points $(0, 0)$ and $(6, 3)$.

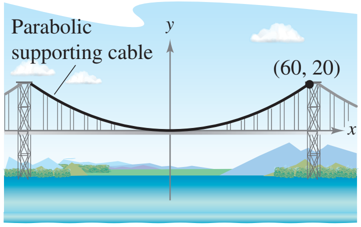
68. Proof

(a) Prove that if any two tangent lines to a parabola intersect at right angles, their point of intersection must lie on the directrix.

(b) Demonstrate the result of part (a) by showing that the tangent lines to the parabola $x^2 - 4x - 4y + 8 = 0$ at the points $(-2, 5)$ and $\left(3, \frac{5}{4}\right)$ intersect at right angles, and that the point of intersection lies on the directrix.

69. Investigation Sketch the graphs of $x^2 = 4py$ for $p = \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}$, and 2 on the same coordinate axes. Discuss the change in the graphs as p increases.

70. Bridge Design A cable of a suspension bridge is suspended (in the shape of a parabola) between two towers that are 120 meters apart and 20 meters above the roadway (see figure). The cable touches the roadway midway between the towers.

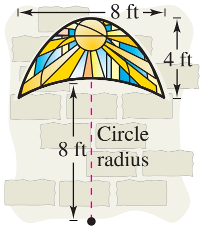


(a) Find an equation for the parabolic shape of the cable.

(b) Find the length of the parabolic cable.

71. Architecture

A church window is bounded above by a parabola and below by the arc of a circle (see figure). Find the surface area of the window.



72. Surface Area A satellite signal receiving dish is formed by revolving the parabola given by $x^2 = 20y$ about the y-axis. The radius of the dish is r feet. Verify that the surface area of the dish is given by

$$2\pi\int_0^r x\sqrt{1 + \left(\frac{x}{10}\right)^2}\,dx = \frac{\pi}{15}[(100 + r^2)^{3/2} - 1000].$$

73. Orbit of Earth Earth moves in an elliptical orbit with the sun at one of the foci. The length of half of the major axis is 149,598,000 kilometers, and the eccentricity is 0.0167. Find the minimum distance (*perihelion*) and the maximum distance (*aphelion*) of Earth from the sun.

74. Satellite Orbit The *apogee* (the point in orbit farthest from Earth) and the *perigee* (the point in orbit closest to Earth) of an elliptical orbit of an Earth satellite are given by A and P. Show that the eccentricity of the orbit is

$$e = \frac{A - P}{A + P}.$$

75. Explorer 18 On November 27, 1963, the United States launched the research satellite Explorer 18. Its low and high points above the surface of Earth were 119 miles and 123,000 miles. Find the eccentricity of its elliptical orbit.

76. Explorer 55 On November 20, 1975, the United States launched the research satellite Explorer 55. Its low and high points above the surface of Earth were 96 miles and 1865 miles. Find the eccentricity of its elliptical orbit.

77. Halley's Comet

Probably the most famous of all comets, Halley's comet, has an elliptical orbit with the sun at one focus. Its maximum distance from the sun is approximately 35.29 AU (1 astronomical unit is approximately 92.956×10^6 miles), and its minimum distance is approximately 0.59 AU. Find the eccentricity of the orbit.

78. Particle Motion Consider a particle traveling clockwise on the elliptical path

$$\frac{x^2}{100} + \frac{y^2}{25} = 1.$$

The particle leaves the orbit at the point $(-8, 3)$ and travels in a straight line tangent to the ellipse. At what point will the particle cross the y-axis?

Area, Volume, and Surface Area **In Exercises 79 and 80, find (a) the area of the region bounded by the ellipse, (b) the volume and surface area of the solid generated by revolving the region about its major axis (prolate spheroid), and (c) the volume and surface area of the solid generated by revolving the region about its minor axis (oblate spheroid).**

79. $\frac{x^2}{4} + \frac{y^2}{1} = 1$

80. $\frac{x^2}{16} + \frac{y^2}{9} = 1$

81. Arc Length Use the integration capabilities of a graphing utility to approximate to two-decimal-place accuracy the elliptical integral representing the circumference of the ellipse

$$\frac{x^2}{25} + \frac{y^2}{49} = 1.$$

82. Conjecture

(a) Show that the equation of an ellipse can be written as

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{a^2(1 - e^2)} = 1.$$

(b) Use a graphing utility to graph the ellipse

$$\frac{(x - 2)^2}{4} + \frac{(y - 3)^2}{4(1 - e^2)} = 1$$

for $e = 0.95$, $e = 0.75$, $e = 0.5$, $e = 0.25$, and $e = 0$.

(c) Use the results of part (b) to make a conjecture about the change in the shape of the ellipse as e approaches 0.

83. Geometry The area of the ellipse in the figure is twice the area of the circle. What is the length of the major axis?

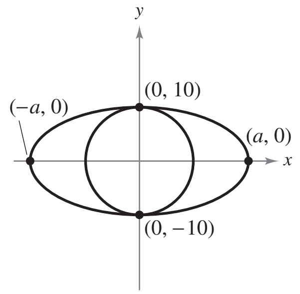


Figure for 83

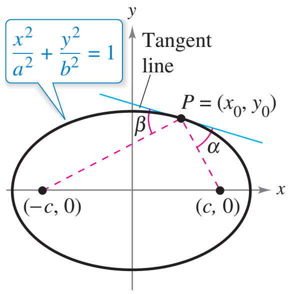


Figure for 84

84. Proof Prove Theorem 10.4 by showing that the tangent line to an ellipse at a point P makes equal angles with lines through P and the foci (see figure). [*Hint:* (1) Find the slope of the tangent line at P, (2) find the slopes of the lines through P and each focus, and (3) use the formula for the tangent of the angle between two lines.]

85. Finding an Equation of a Hyperbola Find an equation of the hyperbola such that for any point on the hyperbola, the difference between its distances from the points $(2, 2)$ and $(10, 2)$ is 6.

86. Hyperbola Consider a hyperbola centered at the origin with a horizontal transverse axis. Use the definition of a hyperbola to derive its standard form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

87. Navigation LORAN (long distance radio navigation) for aircraft and ships uses synchronized pulses transmitted by widely separated transmitting stations. These pulses travel at the speed of light (186,000 miles per second). The difference in the times of arrival of these pulses at an aircraft or ship is constant on a hyperbola having the transmitting stations as foci. Assume that two stations, 300 miles apart, are positioned on a rectangular coordinate system at $(-150, 0)$ and $(150, 0)$ and that a ship is traveling on a path with coordinates $(x, 75)$ (see figure). Find the x-coordinate of the position of the ship if the time difference between the pulses from the transmitting stations is 1000 microseconds (0.001 second).

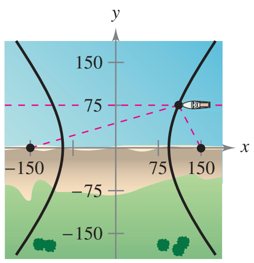


Figure for 87

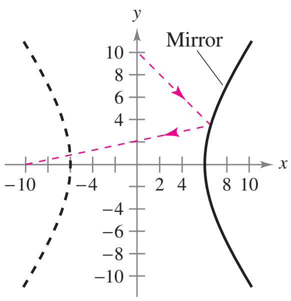


Figure for 88

88. Hyperbolic Mirror A hyperbolic mirror (used in some telescopes) has the property that a light ray directed at the focus will be reflected to the other focus. The mirror in the figure has the equation $(x^2/36) - (y^2/64) = 1$. At which point on the mirror will light from the point $(0, 10)$ be reflected to the other focus?

89. Tangent Line Show that the equation of the tangent line to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_0, y_0) is $\left(\frac{x_0}{a^2}\right)x - \left(\frac{y_0}{b^2}\right)y = 1$.

90. Proof Prove that the graph of the equation

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

is one of the following (except in degenerate cases).

Conic	Condition
(a) Circle	$A = C$
(b) Parabola	$A = 0$ or $C = 0$ (but not both)
(c) Ellipse	$AC > 0$
(d) Hyperbola	$AC < 0$

True or False? **In Exercises 91–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. It is possible for a parabola to intersect its directrix.

92. The point on a parabola closest to its focus is its vertex.

93. If C is the circumference of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b < a$$

then $2\pi b \le C \le 2\pi a$.

94. If $D \ne 0$ or $E \ne 0$, then the graph of $y^2 - x^2 + Dx + Ey = 0$ is a hyperbola.

95. If the asymptotes of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ intersect at right angles, then $a = b$.

96. Every tangent line to a hyperbola intersects the hyperbola only at the point of tangency.

PUTNAM EXAM CHALLENGE

97. For a point P on an ellipse, let d be the distance from the center of the ellipse to the line tangent to the ellipse at P. Prove that $(PF_1)(PF_2)d^2$ is constant as P varies on the ellipse, where PF_1 and PF_2 are the distances from P to the foci F_1 and F_2 of the ellipse.

98. Find the minimum value of

$$(u - v)^2 + \left(\sqrt{2 - u^2} - \frac{9}{v}\right)^2$$

for $0 < u < \sqrt{2}$ and $v > 0$.